Worcester County Mathematics League

Varsity Meet 2
November 30, 2016

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 30th, 2016

Round 1: Fractions, Decimals and Percents

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Solve: $\frac{\frac{x}{3}+\frac{1}{6}}{\frac{1}{2}} = \frac{x+1}{2}$.

2. Find the exact sum as an improper fraction in simplest form:
$$0.\overline{45}+\frac{4}{9}+\frac{2}{3}+0.\overline{7}$$

3. If $x=49$, find 15% of $\frac{2(x+3)}{3}-\frac{4(2x+5)}{5}+\frac{16}{15}$.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 30th, 2016

Round 2: Algebra 1

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Cathy has exactly $2 in a combination of nickels, dimes and quarters. If she has the same number of each coin, how many total coins does Cathy have?

2. Simplify: $\left(\frac{1}{2}-\frac{2}{3}\right)\times\left(\frac{3}{4}-\frac{4}{5}\right)\div\left(\frac{5}{6}-\frac{6}{7}\right)$

3. A project manager needs to complete a job in 30 days. Team A can finish the job in 72 days. Team B can finish the job in 36 days. The manager decides to have both teams work together. What is the least number of days that Team B needs to work in order for the job to be completed exactly on time?

ANSWERS

(1 pt.) 1. ______________________ coins

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________ days

WORCESTER COUNTY MATHEMATICS LEAGUE

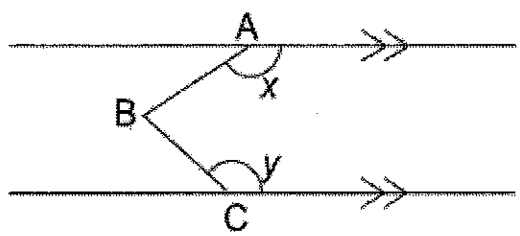
Varsity Meet 2 - November 30th, 2016

Round 3: Parallel Lines and Polygons

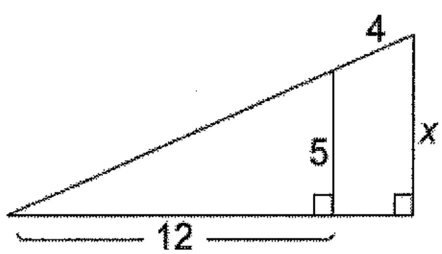
All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. In the figure, if $x = 135°$ and $y = 147°$, find the measure of angle ABC.



2. Find x:



3. A square of side x is inscribed in an equilateral triangle. Determine the height of the triangle in terms of x.

ANSWERS

(1 pt.) 1. ________________ degrees

(2 pts.) 2. ________________

(3 pts.) 3. ________________ square units

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 30th, 2016

Round 4: Sequences and Series

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $t_{n+1} = 2t_n - 3$ and $t_7 = 19$, what is t_3?

2. Find the value of x if it is known that the sequence $2x-1,\ 5x-3,\ 4x+3$ is an arithmetic progression.

3. Determine the 50th term in the sequence: 6, 10, 15, 21, 28, ...

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 30th, 2016
Round 5: Matrices and Systems of Equations

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, where $i^2 = -1$, compute A^5.

2. Solve for x: $\begin{vmatrix} 4 & 1 & -3 \\ x & 5 & -1 \\ 0 & 7 & 4 \end{vmatrix} = 18$

3. If $A = \begin{bmatrix} 9 & -12 \\ 15 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, solve for the matrix X if $A + BX = C^2$.

ANSWERS

(1 pt.) 1. $\begin{bmatrix} & \\ & \end{bmatrix}$

(2 pts.) 2. ______________________

(3 pts.) 3. $\begin{bmatrix} & \\ & \end{bmatrix}$

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 30th, 2016

Team Round

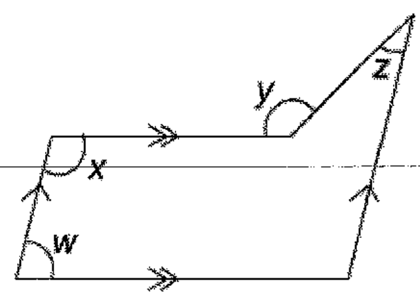
All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 points each)

APPROVED CALCULATORS ALLOWED

1. George has 10 gallons of a solution which is 15% alcohol by volume. How many gallons of the original solution must be replaced by an 80% alcohol solution in order to make a 10 gallon solution which is 70% alcohol by volume?

2. A milkman finishes his route, which is 15 miles long, at 11 am each day. If he were to increase his average rate of travel by a half mile per hour, he would finish his route at 10 am. At what time does the milkman begin his route each day?

3. Find angle w if angle $y = 105^\circ$ and if angle x is twice the measure of angle z.



4. What is the sum of the first hundred elements of the sequence 15, 18, 21, 24, ...?

5. Find the area of the triangle that has vertices at (-2, 2), (1, 5) and (6, -1).

6. When 2017^{2017} is simplified, what is its units digit?

7. If $a = log_8 225$ and $b = log_2 15$, then solve for a in terms of b.

8. It cost $36 to throw a party and each attendee split the cost evenly. If six fewer people had attended, it would have cost each person an additional 50 cents. How many people attended the party?

9. Solve for x : $\frac{3}{x^2+3x} + \frac{x+2}{x+3} = \frac{1}{x}$.

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 30th, 2016
Team Round Answer Sheet

1. ________________________ gallons

2. ________________________

3. ________________________ degrees

4. ________________________

5. ________________________ square units

6. ________________________

7. a = ____________________

8. ________________________ people

9. ________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 30, 2016 ANSWER KEY

Round 1:

1. 1 (Bromfield)
2. $\frac{232}{99}$ or $2.\overline{34}$ (Quaboag)
3. -7 (Auburn)

$\frac{2x+\sqrt{3}x}{2}$

Round 2:

1. 15 (Burncoat)
2. $\frac{-7}{20}$ or -0.35 (Wachusett)
3. 21 (Worcester Academy)

Round 3:

1. 78 (Shepherd Hill)
2. $6\frac{7}{13}$ or $\frac{85}{13}$ (Clinton)
3. $x+\frac{\sqrt{3}}{2}x$ or $x\left(1+\frac{\sqrt{3}}{2}\right)$ (Burncoat)

Round 4:

1. 4 (Auburn)
2. 2 (St. Peter Marian)
3. 1378 (Tahanto)

Round 5:

1. $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (St. Peter-Marian)
2. $\frac{18}{5}$ or $3\frac{3}{5}$ or 3.6 (NDA)
3. $\begin{bmatrix} -1 & 6 \\ 1 & 2 \end{bmatrix}$ (Quaboag)

TEAM Round

1. 8.462 or $8\frac{6}{13}$ or $\frac{110}{13}$ (Shepherd Hill)
2. 5 am (Westborough)
3. 110 (Notre Dame Academy)
4. 16350 (Quaboag)
5. 14 (Algonquin)
6. 7 (St. Johns)
7. $\frac{2b}{3}$ (Worcester Academy)
8. 24 (Bancroft)
9. -1 (Shrewsbury)

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 30th, 2016 - SOLUTIONS

Round 1: Fractions, Decimals and Percents

1. Solve: $\frac{\frac{x}{3}+\frac{1}{6}}{\frac{1}{2}} = \frac{x+1}{2}$.

Solution 1: Begin by removing the denominator on the left side of the equation.

$$\frac{\frac{x}{3}+\frac{1}{6}}{\frac{1}{2}} = \frac{x+1}{2}$$
$$\frac{x}{3}+\frac{1}{6} = \frac{1}{2}\times\frac{x+1}{2}$$
$$\frac{x}{3}+\frac{1}{6} = \frac{x+1}{4}$$
$$\frac{2x+1}{6} = \frac{x+1}{4}$$
$$4\times(2x+1) = 6\times(x+1)$$
$$8x+4 = 6x+6$$
$$2x = 2$$
$$x = 1.$$

Solution 2: We have that

$$\frac{\frac{x}{3}+\frac{1}{6}}{\frac{1}{2}} = \frac{x+1}{2}$$
$$\frac{\frac{2x}{6}+\frac{1}{6}}{\frac{3}{6}} = \frac{x+1}{2}$$
$$\frac{2x+1}{3} = \frac{x+1}{2}$$
$$2(2x+1) = 3(x+1)$$
$$4x+2 = 3x+3$$
$$x = 1.$$

2. Find the exact sum as an improper fraction in simplest form:

$$0.\overline{45}+\tfrac{4}{9}+\tfrac{2}{3}+0.\overline{7}$$

Solution: Begin by converting the repeating decimals into fractions. Let $x = 0.\overline{45}$. Then we have that

$$100x = 45.\overline{45}$$
$$100x - x = 45.\overline{45} - x$$
$$99x = 45.\overline{45} - 0.\overline{45}$$
$$99x = 45$$
$$x = \tfrac{45}{99}$$

Likewise if we let $y = 0.\overline{7}$, we have that

$$\begin{aligned} 10y &= 7.\overline{7} \\ 10y - y &= 7.\overline{7} - y \\ 9y &= 7.\overline{7} - 0.\overline{7} \\ 9y &= 7 \\ y &= \tfrac{7}{9} \end{aligned}$$

Now we have that

$$0.\overline{45} + \tfrac{4}{9} + \tfrac{2}{3} + 0.\overline{7}$$
$$\frac{45}{99} + \frac{4}{9} + \frac{2}{3} + \frac{7}{9}$$
$$\frac{45}{99} + \frac{44}{99} + \frac{66}{99} + \frac{77}{99} = \frac{232}{99}.$$

3. If $x = 49$, find 15% of $\frac{2(x+3)}{3} - \frac{4(2x+5)}{5} + \frac{16}{15}$.

Solution 1: Simplify the expression first and then substitute in the value the x.

$$\frac{2(x+3)}{3} - \frac{4(2x+5)}{5} + \frac{16}{15}$$
$$\frac{10(x+3)}{15} - \frac{12(2x+5)}{15} + \frac{16}{15}$$
$$\frac{10x+30-24x-60+16}{15}$$
$$\frac{-14x-14}{15} = \frac{-14(x+1)}{15} = \frac{-14(50)}{15}$$

We compute 15% of this expression as

$$\frac{15}{100} \times \frac{-14(50)}{15} = \frac{-14(50)}{100} = \frac{-14}{2} = -7.$$

Solution 2: Begin by plugging in the given value for x :

$$\frac{15}{100}\left(\frac{2(x+3)}{3} - \frac{4(2x+5)}{5} + \frac{16}{15}\right)$$
$$\frac{15}{100}\left(\frac{2(49+3)}{3} - \frac{4(2(49)+5)}{5} + \frac{16}{15}\right)$$
$$\frac{15}{100}\left(\frac{5\times2(49+3)}{5\times3} - \frac{3\times4(2(49)+5)}{3\times5} + \frac{16}{15}\right)$$
$$\frac{1}{100}\left(\frac{520}{1} - \frac{1236}{1} + \frac{16}{1}\right)$$
$$\frac{-700}{100} = -7$$

Round 2: Algebra 1

1. Cathy has exactly $2 in a combination of nickels, dimes and quarters. If she has the same number of each coin, how many total coins does Cathy have?

Solution: Let x be the number of each coin that Cathy has. We therefore have that

$$5x + 10x + 25x = 200$$
$$40x = 200$$
$$x = 5.$$

Since Cathy has three kinds of coins, we therefore have that Cathy has 15 coins in total.

2. Simplify: $\left(\frac{1}{2} - \frac{2}{3}\right) \times \left(\frac{3}{4} - \frac{4}{5}\right) \div \left(\frac{5}{6} - \frac{6}{7}\right)$

Solution:

$$\left(\frac{1}{2} - \frac{2}{3}\right) \times \left(\frac{3}{4} - \frac{4}{5}\right) \div \left(\frac{5}{6} - \frac{6}{7}\right)$$
$$\left(\frac{3}{6} - \frac{4}{6}\right) \times \left(\frac{15}{20} - \frac{16}{20}\right) \div \left(\frac{35}{42} - \frac{36}{42}\right)$$
$$\left(-\frac{1}{6}\right) \times \left(-\frac{1}{20}\right) \div \left(-\frac{1}{42}\right)$$
$$\left(-\frac{1}{6}\right) \times \left(-\frac{1}{20}\right) \times \left(-\frac{42}{1}\right)$$
$$\frac{1}{120} \times \frac{-42}{1}$$
$$-\frac{42}{120} = -\frac{14}{40} = -\frac{7}{20}.$$

3. A project manager needs to complete a job in 30 days. Team A can finish the job in 72 days. Team B can finish the job in 36 days. The manager decides to have both teams work together. What is the least number of days that Team B needs to work in order for the job to be completed exactly on time?

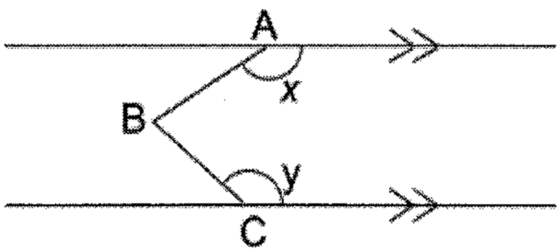
Solution 1: We are given that the job needs to be completed exactly on time, which is 30 days. Therefore, in order to minimize the number of days that Team B works we know Team A is going to work the full thirty days. Let x denote the minimal number of days Team B is needs to work. We have that

$$\tfrac{1}{72} \times 30 + \tfrac{1}{36} \times x = 1$$
$$\tfrac{30}{72} + \tfrac{x}{36} = 1$$
$$30 + 2x = 72$$
$$2x = 42$$
$$x = 21.$$

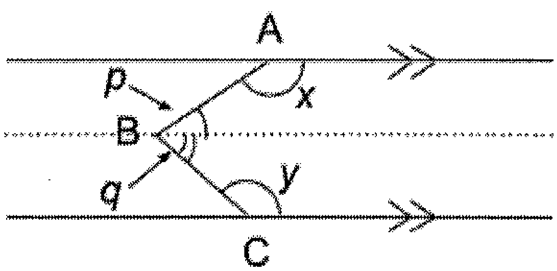
Solution 2: We are given that Team A completes $\frac{1}{72}$ job per day of work and that team B completes $\frac{1}{36}$ job per day of work. In order to minimize the amount of work that Team B needs to do, we know that Team A will work all 30 days, thereby completing $30 \times \frac{1}{72} = \frac{15}{36}$ of the job. If Team A completes $\frac{15}{36}$ of the job, then Team B needs to complete the other $\frac{21}{36}$ of the job. Since Team B works at a rate of $\frac{1}{36}$ job per day of work, we know that this will take them precisely 21 days to accomplish.

Round 3: Parallel Lines and Polygons

1. In the figure, if $x = 135^\circ$ and $y = 147^\circ$, find the measure of angle ABC.



Solution 1: Begin by drawing in a third horizontal line through Point B which is parallel to the two other horizontal lines.



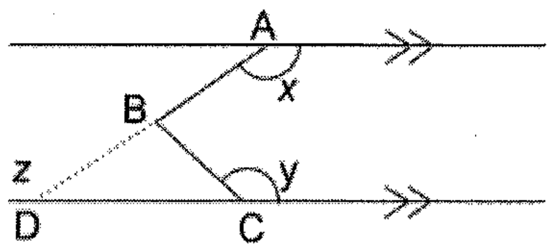
Notice that angle XYZ is precisely equal to $p+q$. To determine the values of p and q, we use the concept of alternate interior angles.

The alternate interior angle to p is precisely $180^\circ - x$, which means that $p = 180^\circ - x = 180^\circ - 135^\circ = 45^\circ$.

The alternate interior angle to q is precisely $180^\circ - y$, which means that $q = 180^\circ - y = 180^\circ - 147^\circ = 33^\circ$.

Therefore, angle XYZ is equal to $45^\circ + 33^\circ = 78^\circ$.

Solution 2: Begin by continuing the line segment AB until it intersects the lower line. Call this point of intersection D.

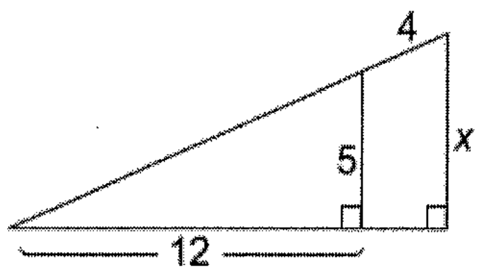


Notice that by the principle of alternate interior angles, angle z shown in the diagram to the left must be equal to angle x. Therefore, we know that angle BDC is equal to $180^\circ - x = 45^\circ$.

We can also deduce that angle DCB is equal to $180^\circ - y = 33^\circ$.

Since the angles within triangle BCD must add up to 180°, we have that angle CBD = $180^\circ - 45^\circ - 33^\circ = 102^\circ$. Therefore, we can conclude that angle ABC = $180^\circ - 102^\circ = 78^\circ$.

2. Find x:



Solution 1: Let y be the hypotenuse of the triangle with legs 12 and 5. By the Pythagorean Theorem, we have that

$$y^2 = 12^2 + 5^2$$
$$y^2 = 169$$
$$y = 13.$$

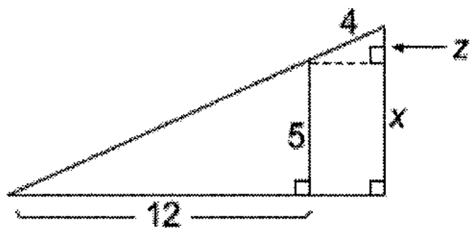
Therefore we know that the hypotenuse of the large triangle is equal to 13 + 4 = 17. Since the large and small triangles in the figure are similar, we have that

$$\frac{5}{13} = \frac{x}{17}$$
$$x = \frac{17\times 5}{13} = \frac{85}{13}.$$

Solution 2: Let y be the hypotenuse of the triangle with legs 12 and 5. By the Pythagorean Theorem, we have that

$$y^2 = 12^2 + 5^2$$
$$y^2 = 169$$
$$y = 13.$$

Now examine the following triangle formed in the top right corner:



Since the base of the small triangle is parallel to the base of the large triangle, we know that the two triangles are similar. Therefore, we have that

$$\frac{5}{13} = \frac{z}{4}$$
$$z = \frac{20}{13}$$

Notice finally that $x = 5 + z = 5 + \frac{20}{13} = \frac{85}{13}$.

3. A square of side x is inscribed in an equilateral triangle. Determine the height of the triangle in terms of x.

Solution: In Figure 1 we can see the described figure. Notice first that the small shaded triangle is yet another equilateral triangle. Since one of the shaded triangle's legs is a side of the square, we know each of the triangle's legs must be equal to x. Since the height of the square is x, we need only compute the height of the shaded triangle in order to know the height of the large triangle.

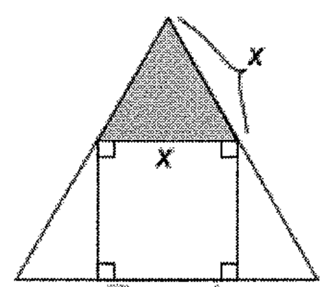


Figure 1

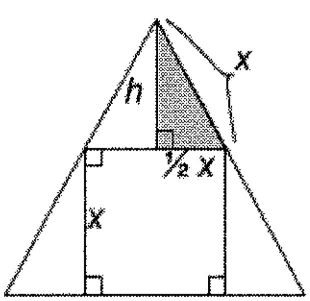


Figure 2

From Figure 2, we can compute the height h by using the Pythagorean Theorem:

$$h^2 = x^2 - \left(\tfrac{1}{2}x\right)^2 = \tfrac{3}{4}x^2$$
$$h = \tfrac{\sqrt{3}}{2}x.$$

Therefore, we have that the large triangle's height is $x + \frac{\sqrt{3}}{2}x = x\left(1 + \frac{\sqrt{3}}{2}\right)$.

Round 4: Sequences and Series

1. If $t_{n+1} = 2t_n - 3$ and $t_7 = 19$, what is t_3?

Solution 1: Use the recursive formula to solve for t_n in terms of t_{n+1}:

$$t_{n+1} = 2t_n - 3$$
$$t_n = \frac{t_{n+1}+3}{2}$$

Now we use this formula to work backwards to t_3.

$$t_6 = \frac{t_7+3}{2} = \frac{19+3}{2} = 11.$$
$$t_5 = \frac{t_6+3}{2} = \frac{11+3}{2} = 7.$$
$$t_4 = \frac{t_5+3}{2} = \frac{7+3}{2} = 5.$$
$$t_3 = \frac{t_4+3}{2} = \frac{5+3}{2} = 4.$$

Solution 2: Since we are given that $t_7 = 19$, we have that

$$t_{6+1} = 2t_6 - 3 = 19 \quad \rightarrow \quad t_6 = \tfrac{19+3}{2} = 11$$

$$t_{5+1} = 2t_5 - 3 = 11 \quad \rightarrow \quad t_5 = \tfrac{11+3}{2} = 7$$

$$t_{4+1} = 2t_4 - 3 = 7 \quad \rightarrow \quad t_4 = \tfrac{7+3}{2} = 5$$

$$t_{3+1} = 2t_3 - 3 = 7 \quad \rightarrow \quad t_3 = \tfrac{5+3}{2} = 4.$$

2. Find the value of x if it is known that the sequence $2x-1,\ 5x-3,\ 4x+3$ is an arithmetic progression.

Solution: We know that in an arithmetic sequence the difference between two consecutive elements is always a constant number, which we will call y.

In this problem we have that

$$(5x-3)-(2x-1) = y \qquad (1)$$
$$(4x+3)-(5x-3) = y \qquad (2)$$

Since equations (1) and (2) have the same right hand side, we can equate their left hand sides to one another.

$$(5x-3)-(2x-1) = (4x+3)-(5x-3)$$
$$5x-3-2x+1 = 4x+3-5x+3$$
$$3x-2 = 6-x$$
$$4x = 8$$
$$x = 2.$$

3. Determine the 50th term in the sequence: 6, 10, 15, 21, 28, ...

Solution 1: Begin by determining a general form for the sequence. We have that

$$a_n = a_{n-1} + (n+2) \text{ where } a_1 = 6.$$

We are interested in the fiftieth term in the sequence which can be written as

$$a_{50} = a_{49} + 52$$
$$a_{50} = (a_{48} + 51) + 52$$
$$a_{50} = (a_{47} + 50) + 51 + 52$$
$$\dots$$
$$a_{50} = a_1 + 4 + 5 + 6 + \dots + 51 + 52$$

The sum from 4 to 52 is given by

$$\tfrac{(52)(53)}{2} - 1 - 2 - 3$$
$$\tfrac{2756}{2} - 1 - 2 - 3$$
$$1378 - 6.$$

Therefore, we have that $a_{50} = 6 + (1378 - 6) = 1378.$

Solution 2: Notice that the given sequence is a subsequence of the Triangular Numbers, which are 0, 1, 3, 6, 10, 15, ... The formula for the n^{th} Triangular number is

$$T_n = \tfrac{(n)(n-1)}{2}$$

Since the first number in the given sequence is T_4, we know that the 50th term in the given sequence will be T_{53}. We have that

$$T_{53} = \tfrac{(53)(52)}{2} = 1378.$$

Round 5: Matrices and Systems of Equations

1. If $A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, where $i^2 = -1$, compute A^5.

Solution: We know that $A^5 = (A^2)(A^2)(A)$. We have that

$$A^2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \times \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0-1 & 0+0 \\ 0+0 & 0-1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Notice that this is just the 2 x 2 identity matrix I_2 multiplied by the scalar -1. Therefore, we have that $(A^2)(A^2)$ is equal to $(-1I_2)(-1I_2) = (-1)^2 I_2 I_2 = I_2$.

Finally since $A^5 = (A^2)(A^2)(A) = I_2 A = A$. Therefore, $A^5 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

2. Solve for x: $\begin{vmatrix} 4 & 1 & -3 \\ x & 5 & -1 \\ 0 & 7 & 4 \end{vmatrix} = 18$

Solution 1: Writing out the determinant of the matrix gives us the equation

$$(4)(5)(4) + (1)(-1)(0) + (-3)(x)(7) - (0)(5)(-3) - (7)(-1)(4) - (4)(x)(1) = 18$$

$$80 - 21x + 28 - 4x = 18$$

$$90 = 25x$$

$$x = \tfrac{90}{25}$$

Solution 2: An alternative method of writing the determinant of the matrix is

$$4[(5 \times 4) - (-1 \times 7)] - x[(1 \times 4) - (-3 \times 7)] + 0[(-1 \times 1) - (-3 \times 5)] = 18$$

$$4[20 + 7] - x[4 + 21] + 0 = 18$$

$$108 - 25x = 18$$

$$90 = 25x$$

$$x = \tfrac{90}{25}$$

3. If $A = \begin{bmatrix} 9 & -12 \\ 15 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, solve for the matrix X

if $A + BX = C^2$.

Solution: Notice first that since the determinant of B is -22, we know that it is invertible. Therefore, we can algebraically solve for X :

$$A + BX = C^2$$
$$BX = C^2 - A$$
$$B^{-1}BX = B^{-1}(C^2 - A)$$
$$IX = B^{-1}(C^2 - A)$$
$$X = B^{-1}(C^2 - A)$$

Now we can compute all the necessary terms.

$$C^2 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$
$$C^2 - A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} - \begin{bmatrix} 9 & -12 \\ 15 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 18 \\ -9 & 14 \end{bmatrix}$$
$$B^{-1} = \frac{1}{-22} \begin{bmatrix} -5 & -3 \\ -4 & 2 \end{bmatrix}$$

Therefore, we have

$$X = \frac{1}{-22} \begin{bmatrix} -5 & -3 \\ -4 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 18 \\ -9 & 14 \end{bmatrix} = \frac{1}{-22} \begin{bmatrix} 22 & -132 \\ -22 & -44 \end{bmatrix} = \begin{bmatrix} -1 & 6 \\ 1 & 2 \end{bmatrix}$$

Team Round

1. George has 10 gallons of a solution which is 15% alcohol by volume. How many gallons of the original solution must be replaced by an 80% alcohol solution in order to make a 10 gallon solution which is 70% alcohol by volume?

Solution 1: Let x be the amount of the original liquid that we need to replace in order to create the desired concentration. We have that

$$(10 - x)(0.15) + x(0.8) = 10(0.7)$$
$$1.5 - 0.15x + 0.8x = 7$$
$$0.65x = 5.5$$
$$x = 8.462.$$

Solution 2: Let x be the number of gallons of 80% solution needed. Since we need a total of 10 gallons of final solution, we know that we will have $10-x$ gallons of 15% solution. Moreover, since the final solution is 10 gallons at 70% alcohol concentration, we ultimately need 7 gallons of alcohol. Therefore, we have that

$$0.8x + 0.15(10-x) = 7$$
$$1.5 - 0.15x + 0.8x = 7$$
$$0.65x = 5.5$$
$$x = 8.462.$$

2. A milkman finishes his route, which is 15 miles long, at 11 am each day. If he were to increase his average rate of travel by a half mile per hour, he would finish his route at 10 am. At what time does the milkman begin his route each day?

Solution 1: Let x denote the milkman's original average speed. Therefore, it takes the milkman $\frac{15}{x}$ hours to complete his normal route. When the milkman increases his speed by one half mile per hour, we are given that he completes his route in $\frac{15}{x}-1$ hours.

Therefore, if we multiply $\frac{15}{x}-1$ hours by the milkman's new increased average speed, we should get a product of precisely 15 miles. That is,

$$(\tfrac{15}{x}-1)(x+0.5) = 15$$
$$15+\tfrac{15}{2x}-x-0.5 = 15$$
$$30x+15-2x^2-x = 30x$$
$$-2x^2-x+15=0$$
$$(-2x+5)(x+3)$$
$$x=2.5 \text{ or } -3.$$

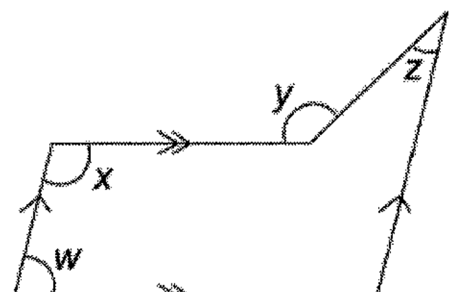
Since the milkman has a positive speed, we know that $x = 2.5$. Since $\frac{15}{2.5} = 6$, we have that the milkman takes six hours to complete his route at normal speed, which means he starts his route at 5 am.

Solution 2 (Guess and Check): We know that Time = Distance/Speed. Therefore, since we know the distance is 15 miles, we can guess different speeds and see if we can find two speeds that yield a difference in time of precisely one hour:

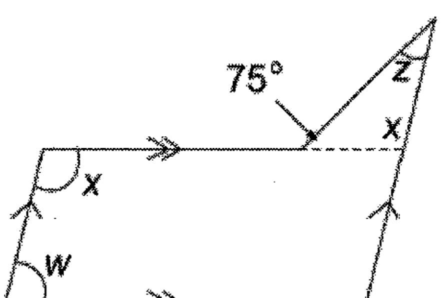
Distance (miles)	Speed (mph)	Time (hours)
15	5	3
	4	3.75
	3.5	4 2/7
	3	**5**
	2.5	**6**
	2	7.5
	1	15

We notice that the speeds 3 mph and 2.5 mph yield a difference in travel time of precisely one hour, which means that the milkman's original speed was 2.5 mph. Since the milkman at his original speed ends his route at 11 am and takes 6 hours to finish, we know he starts his route at 5 am.

3. Find angle w if angle $y = 105^\circ$ and if angle x is twice the measure of angle z.



Solution:



Begin by focusing on the small triangle in the top right corner of the figure. We use alternate interior angles to see that the lower right corner of the small triangle is equal to angle x.

Moreover, since angle $y = 105^\circ$, we know by supplementary angles that the lower left corner of the small triangle must have measure 75°.

We are given that angle x is twice the measure of angle z, and we know that the sum of the angles in the small triangle is 180°. Therefore, we have that

$$x + z + 75^\circ = 180^\circ \qquad (1)$$
$$z = \tfrac{1}{2}x \qquad (2)$$

Substituting equation (2) into equation (1) gives

$$x + z + 75^\circ = 180^\circ$$
$$x + \tfrac{1}{2}x + 75^\circ = 180^\circ$$
$$\tfrac{3}{2}x = 105^\circ$$
$$x = 70^\circ$$

Finally, since angles x and w are supplementary, we have that

$$x + w = 180^\circ$$
$$70^\circ + w = 180^\circ$$
$$w = 110^\circ$$

4. What is the sum of the first hundred elements of the sequence 15, 18, 21, 24, ...?

Solution 1: We have that the general term in this sequence can be written as $a_n = (n+4) \times 3$. Therefore, we have that

$$\sum_{n=1}^{100} a_n$$
$$\sum_{n=1}^{100} (n+4) \times 3$$
$$3 \sum_{n=1}^{100} n + 4$$
$$3 \times \left(4 \times 100 + \sum_{n=1}^{100} n \right)$$
$$3 \times 400 + 3\tfrac{100 \times 101}{2} = 16350.$$

Solution 2 (Gauss's Method): We notice immediately that the difference between consecutive terms in this sequence is always 3. To get the second term, we add 3, or 1×3, to the first term. To get the third term, we add 6, or 2×3, to the first term. Hence to get the hundredth term we add 99×3 or 297 to the first term, which gives 312.

Now that we know the hundredth term, we want to compute the desired sum. To do so, we can follow Gauss's method and arrange two copies of the first hundred terms in two lines, one going forward and the other backward:

	15	18	21	...	306	309	312
	312	309	306	...	21	18	15
+							
	327	327	327	...	327	327	327

Adding the elements of the two lines vertically, we see that each column yields a sum of 327. Therefore, we know that two times the sum of the first hundred terms is equal to $100 \times 327 = 32700$. Finally, we have that the desired sum is simply $32700 \div 2 = 16350$.

5. Find the area of the triangle that has vertices at (-2, 3), (1, 5) and (6, -1).

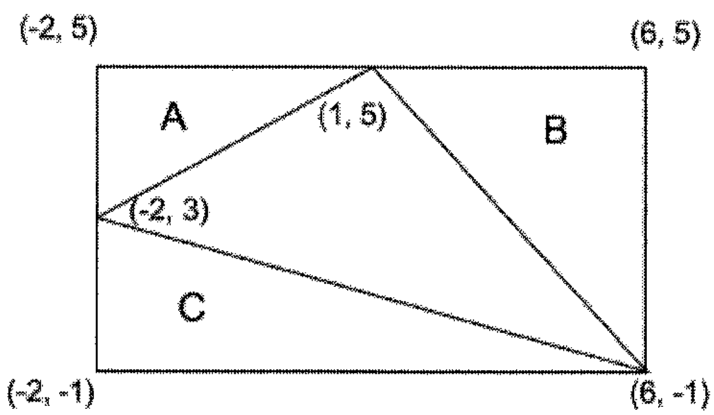
Solution 1 (Matrix Solution): We can find the area of this triangle by taking one half of the absolute value of determinant of the following 3x3 matrix:

$$\begin{bmatrix} -2 & 3 & 1 \\ 1 & 5 & 1 \\ 6 & -1 & 1 \end{bmatrix}$$

Expanding this determinant, we get

$$\text{Area} = \tfrac{1}{2}|(-2)(5)(1) + (3)(1)(6) + (1)(1)(-1) - (6)(5)(1) - (-1)(1)(-2) - (1)(1)(3)|$$
$$\text{Area} = \tfrac{1}{2}|-10 + 18 - 1 - 30 - 2 - 3|$$
$$\text{Area} = \tfrac{1}{2}|-28|$$
$$\text{Area} = 14 \text{ sq units}$$

Solution 2 (Graphing): We can draw a rectangle around the described triangle as seen in the figure below.



Using the coordinates in the figure, we can compute the base and height for each of triangles A, B and C:

Triangle A: $\text{Area} = \frac{1}{2} \times 2 \times 3 = 3$ sq units
Triangle B: $\text{Area} = \frac{1}{2} \times 6 \times 5 = 15$ sq units
Triangle C: $\text{Area} = \frac{1}{2} \times 4 \times 8 = 16$ sq units

Moreover, the area of the entire rectangle is $8 \times 6 = 48$ sq units. Therefore, the desired area is equal to $48 - (3 + 15 + 16) = 48 - 34 = 14$ sq units.

6. When 2017^{2017} is simplified, what is its units digit?

Solution: Since we are only interested in the units digit of the simplified number, we can equivalently examine 7^{2017}. To do this, we will look for patterns in the powers of 7.

$7^1 = 7$	ends in 7
$7^2 = 49$	ends in 9
$7^3 = 343$	ends in 3
$7^4 = 2401$	ends in 1
$7^5 = 16807$	ends in 7

At this point we know that $7^6 = 7 \times 16807$ must end in 9 because 7 times a number that ends in 7 will always end with 9. And the pattern continues so on and so forth repeating after every four powers.

Since 2017 has remainder 1 when divided by 4, we can conclude that 2017^{2017} must end with the digit 7.

7. If $a = log_8 225$ and $b = log_2 15$, then solve for a in terms of b.

Solution: Begin by noticing that $225 = 15^2$. That allows us to write

$$a = log_8 225 = log_8 15^2 = 2 log_8 15.$$

Now we can use the change of base formula to get that

$$a = 2log_8 15 = \frac{2log_2 15}{log_2 8} = \frac{2log_2 15}{3} = \frac{2b}{3}.$$

8. It cost \$36 to throw a party and each attendee split the cost evenly. If six fewer people had attended, it would have cost each person an additional 50 cents. How many people attended the party?

Solution 1: Let x be the the number of people that attended the party. We have that

$$\frac{36}{x} + .5 = \frac{36}{x-6}$$
$$36 + .5x = \frac{36x}{x-6}$$
$$(x-6)(36+.5x) = 36x$$
$$.5x^2 + 36x - 216 - 3x = 36x$$
$$.5x^2 - 3x - 216 = 0$$
$$x^2 - 6x - 432 = 0$$
$$(x-24)(x+18) = 0$$
$$x = 24 \text{ or } -18.$$

Since there are a positive number of people at the party, we conclude that the desired answer is 24 people.

Solution 2 (Guess and check): To solve the problem we will take guesses at the original number of guests, compute the cost per person and likewise the cost if there are 6 fewer guests. We will keep making guesses until the difference between the cost per person and the cost per person with six fewer people is precisely 50 cents.

Original # of Guests	Cost per person	Cost per person (with six fewer)	Difference
9	4	12	8
12	3	6	3
18	2	3	1
24	1.50	2	**.50**

We see that this condition is met with an original number of guests of 24 people.

9. Solve for x : $\frac{3}{x^2+3x} + \frac{x+2}{x+3} = \frac{1}{x}$.

Solution 1: Start by giving every term a common denominator:

$$\frac{3}{x^2+3x} + \frac{x+2}{x+3} = \frac{1}{x}$$

$$\frac{3}{x(x+3)} + \frac{x(x+2)}{x(x+3)} = \frac{x+3}{x(x+3)}$$

Now we have that

$$3 + x(x+2) = x+3$$

$$3 + x^2 + 2x = x + 3$$

$$x^2 + x = 0$$

$$x(x+1) = 0$$

So we have that $x = 0$ or $x = -1$. However, we can eliminate $x = 0$ as a solution since it would lead to a zero denominator in the original equation. Therefore, $x = -1$.

Solution 2: We have that

$$\frac{3}{x^2+3x} + \frac{x+2}{x+3} = \frac{1}{x}$$
$$\frac{3}{x(x+3)} + \frac{x+2}{x+3} = \frac{1}{x}$$

We know that $x \neq 0$ since it would leave the first term in the equation undefined. Therefore, we can multiply the second term by $\frac{x}{x}$:

$$\frac{3}{x(x+3)} + \frac{x+2}{x+3} \times \frac{x}{x} = \frac{1}{x}$$
$$\frac{3+x(x+2)}{x(x+3)} = \frac{1}{x}$$
$$\frac{3+x^2+2x}{(x+3)} = \frac{1}{1}$$
$$3 + x^2 + 2x = x + 3$$
$$x^2 + x = 0$$
$$x(x+1) = 0$$

This means that either $x = 0$ or $x = -1$, but we already ruled out the possibility that $x = 0$, so $x = -1$.